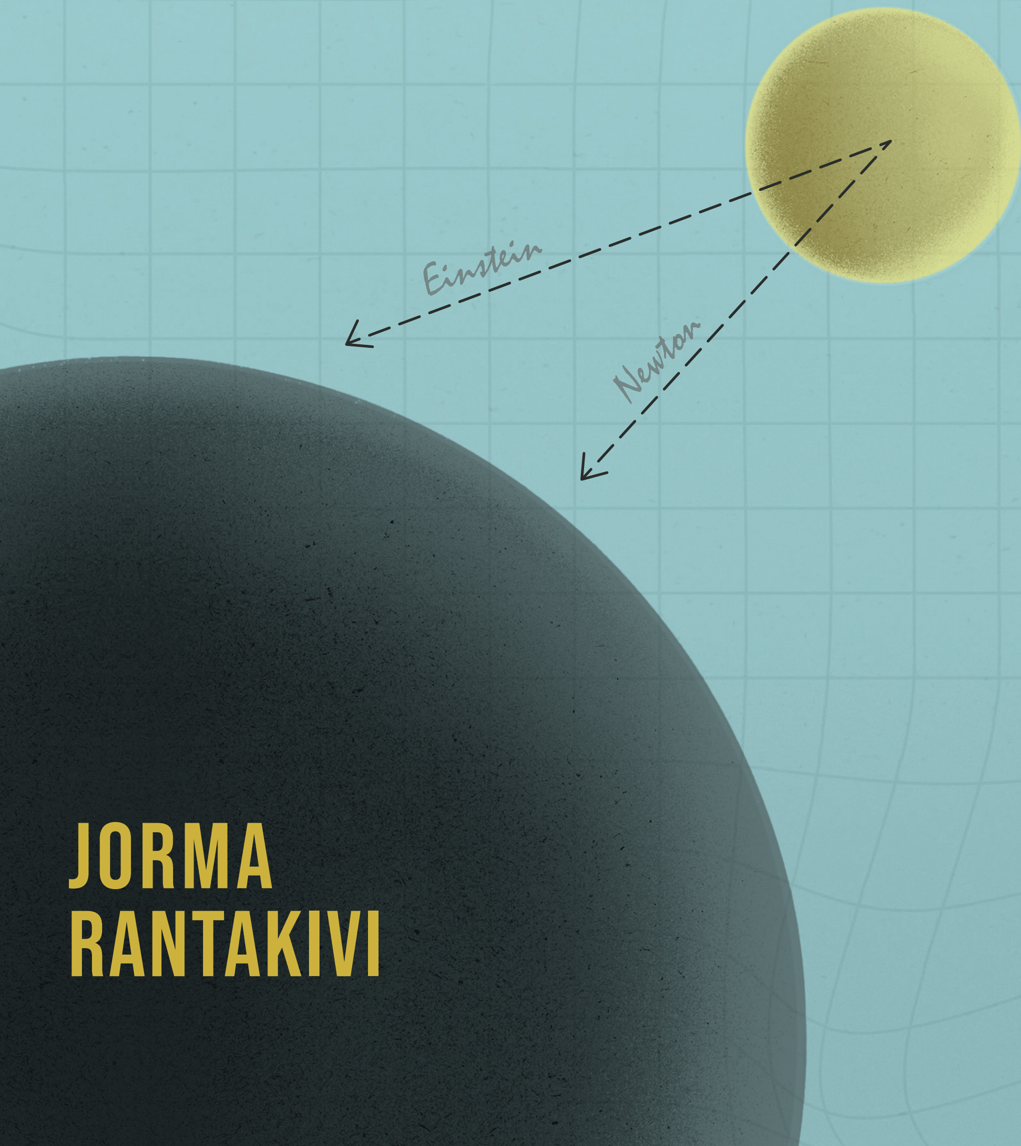
GRAVITATION III

Velocity-dependent force



JORMA RANTAKIVI

Gravitation III – Velocity-dependent force

JORMA RANTAKIVI

Gravitation III – Velocity-dependent force

Layout and cover: Books on Demand
Publisher: BoD · Books on Demand, Mannerheimintie 12 B, 00100 Helsinki, bod@bod.fi
Manufactured: Libri Plureos GmbH, Friedensallee 273, 22763 Hamburg, Germany
ISBN: 978-952-89-3917-7

Contents

Introduction

Maxwell's equations give the electric field and the magnetic field when the charge density and current density of space are known. When a charged particle moves in an electromagnetic field, it is subjected to the Lorenz force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

where **F** is the force acting on the particle, q is the charge of the particle, **E** is the electric field, v is the velocity of the particle, and **B** is the magnetic field. So, the force acting on a charged particle is not only a function of position, but also depends on the velocity of the charged particle. I don't know the history very well, but I understand that in the early 1900s, when the theory of gravity was to be reformed, the model was a field theory similar to Maxwell's equations, in which the gravitational force acting on an object would be a function of the position and velocity of the object. This led to Einstein's field equations, the solution of which gives space the metric tensor. The metric tensor describes the curvature of space. The metric tensor can be used to calculate the length of a curve in curved space. Orbits that minimize curve length are possible orbits of a celestial body in curved space.

The length of the curve can be represented as a definite integral, so the time-dependent position vector that minimizes the length of the curve is obtained as a solution to Euler equations. Euler equations are second-order differential equations for the components x(t), y(t), z(t) of the time-dependent position vector and their first and second derivatives with respect to time. The solution of the differential equation x(t), y(t), z(t) is the orbit of the celestial body in curved space. If you look at the same group of Euler equations from another perspective, you can see a linear group of equations for the components of the acceleration as a function of the position and velocity of the object. Since acceleration is equal to the force per unit mass, Euler's equations give directly the relativistic force acting on a body as a function of the position and velocity of the body. As the book will show below, the situation is not so simple, because the velocity of an object is different relative to each celestial body moving in space, and therefore no one specific velocity can be applied to the object when calculating its orbit. This problem is circumvented in the book by calculating the forces for each pair of objects separately and summing up these forces. In this way, a result is obtained, but the result is probably not correct except when calculating a case where there are only two interacting objects. It is probably a good approximation for the relativistic solution in a weak gravitational field, where the relativistic effects are small, such as when calculating the Sun, Moon, Earth system, for which the relativistic calculation gives almost exactly the same result as a calculation according to Newtonian mechanics.

1. Orbit of a celestial body

1.1 About labelling

This book uses two types of notation for velocities and accelerations for reasons of printing technology and readability

(1.1.1) $$\frac{dx}{dt} = v_x = \dot{x},\ \frac{dy}{dt} = v_y = \dot{y},\ \frac{dz}{dt} = v_z = \dot{z}$$

$$\frac{d^2x}{dt^2} = a_x = \ddot{x},\ \frac{d^2y}{dt^2} = a_y = \ddot{y},\ \frac{d^2z}{dt^2} = a_z = \ddot{z}$$

1.2 Metrics

As shown in reference /1/, the mass distribution of space determines the metric tensor of space $g_{ij}(x,y,z)$. The four-dimensional curve element of the time dependent curve x(t), y(t), z(t) takes the form

(1.2.1)

$$ds = \sqrt{g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2 + g_{12}dxdy + g_{13}dxdz + g_{23}dydz + g_{44}d(ct)^2}$$

$$= \sqrt{g_{11}v_x^2 + g_{22}v_y^2 + g_{33}v_z^2 + g_{12}v_xv_y + g_{13}v_xv_z + g_{23}v_yv_z + g_{44}c^2}dt$$

and the four-dimensional curve length s is given in space-time as an integral

(1.2.2)

$$s = \int_0^t \sqrt{g_{11}v_x^2 + g_{22}v_y^2 + g_{33}v_z^2 + g_{12}v_xv_y + g_{13}v_xv_z + g_{23}v_yv_z + g_{44}c^2}dt$$

In the formulas v_x is the velocity parallel to the x-axis, v_y is the velocity parallel to the y-axis, and v_z is the velocity parallel to the z-axis. In a rectilinear Euclid space $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = 1$ and $g_{12} = g_{13} = g_{23} = 0$.

In general relativity, such an orbit x(t), y(t), z(t) is found that minimizes the orbital length in a space with the metric g. The orbit calculation is divided into two parts. First, we have to solve the metric g of space and then determine the shortest line, i.e. the geodesic line, in this space with some boundary conditions. As a boundary condition you may give e.g. the distance of the planet from the Sun at aphelion and perihelion. This geodesic line is then the orbit of the celestial body.

The solution for the metric in a general case (arbitrary mass distribution in space) is obtained by transforming to the so-called gravitational coordinate system as shown in reference /1/ and briefly recaped below.

1.3 Gravitational coordinate system

Let the density distribution in space be ρ. Then Newton's potential field U realizes

(1.3.1) $$U(x,y,z) = -G \iiint \frac{\rho(x_0,y_0,z_0)}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}} dx_0 dy_0 dz_0$$

(1.3.2) $$\nabla^2 U = 4\pi G\rho$$

In the central field, where an object of mass M is located at the origin, the potential field takes shape

(1.3.3) $$U = -\frac{GM}{r} = -\frac{GM}{\sqrt{x^2+y^2+z^2}}$$

where G is the general gravitational constant.

The potential field of several (n) celestial bodies at a point (x,y,z) is calculated as a sum

(1.3.4) $$U(x,y,z) = \sum_{i=1}^{n} -\frac{GM_i}{r_i},$$

General relativity gives the orbit of a celestial body as a solution to Euler's equations. The solution is a time-dependent position vector that minimizes the orbital length in curved space. The same Euler equations can also be viewed from another perspective. They can be interpreted as a group of equations for components parallel to the coordinate axes of the acceleration of the body as a function of the velocity and position of the body in a rectilinear Euclidean space. Accelerations can be solved numerically from the group of equations. Since the acceleration is the same as the force per unit mass, the result is the relativistic force acting on the object. In this way, the orbits of several bodies can be calculated simultaneously according to general relativity. The book presents a more detailed calculation algorithm, a program in the FORTRAN language and several calculated example cases.

BoD

9 789528 939177